Galois Extensions of Structured Ring Spectra

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Chapter 1

Classical Galois theory

Definition 1.0.1. We recall that for a finite group G, an extension of fields $K \hookrightarrow L$ is G-Galois iff there is an of G on L in the category of fields, and $L^G = K$. Here L^G denote the G-fixed point of L.

Moreover, we see that we have a natural additional condition.

Proposition 1.0.2. *Namely, we consider the map of F-algebra $h : E \otimes_F E \to \prod_G E$ by $(e_1 \otimes e_2) \mapsto (g \mapsto e_1 \cdot g(e_2))$. We claim that this map is an isomorphism by choosing a set of normal basis of F over E.*

We see an amazing (maybe not that amazing) analogy of Galois extensions with the theory of G-principal bundles.

Definition 1.0.3. Recall from last week that we traditianally define a G-principal bundle to be a bundle $p : Y \to X$ with a G action on Y such that $G \times Y = Y \times_X Y$, and $Y/G = X$.

Therefore, by passing to the language of algebraic geometry, we see that $G \curvearrowright L$ induces an action $\mathrm{Spec}L \curvearrowleft G$, and the two conditions translates to $\mathrm{Spec}L/G = \mathrm{Spec}K$, and $\mathrm{Spec}L \times G \simeq \mathrm{Spec}L \times_{\mathrm{Spec}K} \mathrm{Spec}L$.

Thus we may extend Galois theory naturally to the category **Ring**.

Definition 1.0.4. We define for a map $R \to T$, and for a finite group $G \curvearrowright T$ through R**Alg**, the extension is Galois iff we have $R \simeq T^G$ and $T \otimes_R T = \prod_G T$. Equivalently, the extension is Galois iff we have the map $\mathrm{Spec}T \to \mathrm{Spec}R$ is a G-principal bundle.

As an application, we see that we have an unramified G-Galois extension $K \to L$ induces a G-Galois extension on the integral rings.

G-Galois extensions of rings admit nice properties, listed below. We omit the proof since the speaker is not that good on commutative algebra, although the proof may be direct using algebraic geometry.

Lemma 1.0.5. *For $R \to T$ a G-Galois extension, we have T faithfully flat and finitely generated projective over R.*

We immediately see that the above definition generalizes immediately to the rings in the $\mathbf{Sp}_E$, or its subcategories consisting of A-modules (where A is a E-local ring spectra). However, we will work on **Sp** since on most occasions there are completely no obstructions to generalizing the proof to $\mathbf{Sp}_E$. We can make a slight generalization to the above definition by allowing G to be any topological group with some nice properties. This requires some work.

Chapter 2

Preparations on topological groups

Definition 2.0.1. Recall that for a spectrum X, its dual is defined to be $\mathrm{Map}(X, S)$, where S is the sphere spectrum. We define a topological group G to be stably dualizable if its suspension spectrum $S[G] := \Sigma^\infty_+ G$ is dualizable. We denote for simplicity its dual to be DG_+.

Here we explain the notation $S[G]$. We claim that this spectrum is an analogy for the classical group algebra $\mathbb{Z}[G]$ (the latter makes sense only when G is discrete, of course). Namely, we claim the following.

Definition 2.0.2. For a ring spectrum R, a right R-module A and a left R-module B, we define the relative tensor product $A \otimes_R B$ to be the following bar construction

$$\varinjlim\left(A \otimes B \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; A \otimes R \otimes B \qquad \cdots \right)$$

Similarly we define for a left R-module A and a left R-module B the internal hom for modules $\mathrm{Map}_R(A, B)$ as the cobar construction

$$\varprojlim\left(\mathrm{Map}(A, B) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathrm{Map}(A \otimes R, B) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathrm{Map}(A \otimes R \otimes R, B) \qquad \cdots \right)$$

Notice that the second construction indeed respects the R-module structure of B since the map (for example) $\mathrm{Map}(A, B) \to \mathrm{Map}(R \otimes A, B)$ is given by the dual of

$$R \otimes A \otimes \mathrm{Map}(A, B) \to R \otimes B \to B$$

Notice that this construction is equivalent to the mapping spectrum given by the stable structure of R-module categories since they can be identified for A a free module.

Proposition 2.0.3. *$S[G]$ admits a ring spectrum structure, and for a spectrum X, a left $S[G]$-module structure on X is equivalent to a left G-action on X, and if we equip S with the trivial bimodule structure, we have $S \otimes_{S[G]} X = X_{hLG}$, and $\mathrm{Map}_{LS[G]}(S, X) = X^{hLG}$. We have a similar theorem for right $S[G]$-module structures on X. Likewise, we define for an $\mathbb{E}_\infty$-ring spectrum A, the group algebra $A[G] = A \otimes S[G]$.*

Proof. The ring spectrum structure of $S[G]$ comes from the property that Σ^∞_+ is symmetric monoidal, allowing us to inherit a ring structure from the group structure of G.
Then, given a $S[G]$-module structure on X, the $g \in G$ acts on X by the map $X \otimes S \hookrightarrow X \otimes S[G] \to X$, where $S \hookrightarrow S[G]$ is given by the inclusion $\{g\} \hookrightarrow G$, and the second morphism the structure map of the module.
Therefore we have equivalences of categories $\mathrm{Fun}(BG, \mathbf{Sp}) \simeq_{S[G]} \mathbf{Mod}$. Then, notice that given a spectrum X, the trivial G-action on X is given by the module structure $S[G] \otimes X \to X$ given by the projection of all elements of G to a point. In particular, the trivial action of G on S is

given this way. We conclude that this functor is equivalent to associating a spectrum X the constant X-valued BG-diagram in $\mathbf{Sp}$. Moreover, We see that $S \otimes_{S[G]} X$ and $\mathrm{Map}_{LS[G]}(S, X)$ are by definition the left adjoint and right adjoint of the inclusion via trivial modules. By the uniqueness of the left and right adjoint, we see that $S \otimes_{S[G]} X = X_{hLG}$ and $\mathrm{Map}_{LS[G]}(S, X) = X^{hLG}$. □

Corollary 2.0.4. *DG_+ admit a right G-action, with the right $S[G]$-module sturcture given by the dual of the left $S[G]$-module structure on $S[G]$, explicitly given by the following*

$$\mathrm{Hom}_{\mathbf{Sp}}(\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G], \mathrm{Map}_{\mathbf{Sp}}(S[G], S)) \simeq \mathrm{Hom}_{\mathbf{Sp}}(\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G] \otimes S[G], S)$$

and the map is the preimage of $\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes (S[G] \otimes S[G]) \to \mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G] \to S$.

With this proposition, we from now on abbreviate all our notations by replacing $S[G]$ with G (i.e. G-modules, etc).

Construction 2.0.5. *Notice that $S[G]$ admits a Hopf algebra structure with coalgebra structure given by the diagonal map $\Delta : G \to G \times G$, the counit given by the map collapsing G to a point, and the conjugation given by the inverse map $i : G \to G$.*

Definition 2.0.6. For a stable dualizable group, we define the dualizing spectrum $S^{\mathrm{ad}G} = S[G]^{hRG}$, where $S[G]$ is regarded as a right G-module.

Theorem 2.0.7. *We have $DG_+ \otimes S^{\mathrm{ad}G} \simeq S[G]$.*

Proof. We consider the shear map given by

$$DG_+ \otimes S[G] \to DG_+ \otimes S[G] \otimes S[G] \to DG_+ \otimes S[G]$$

where the first map is given by the diagonal map of $S[G]$ and the second map the structure map of DG_+. We claim that this map has an inverse given by

$$DG_+ \otimes S[G] \to DG_+ \otimes S[G] \otimes S[G] \to DG_+ \otimes S[G] \otimes S[G] \to DG_+ \otimes S[G]$$

where the middle map is induced by the inverse map on the middle $S[G]$-factor. Then, notice that if we take a right G-action on the left given by the right G-action on $S[G]$, the shear map takes this G-action to the standard right G-actions. Thus taking homotopy fixed points on both sides, we obtain on the left $(DG_+ \otimes S[G])^{hRG} \simeq DG_+ \otimes S^{\mathrm{ad}G}$, and on the right we have

$$\begin{aligned}
&(DG_+ \otimes S[G])^{hRG} \\
\simeq &\mathrm{Map}_{\mathbf{Sp}}(S[G], S[G])^{hRG} \\
\simeq &\mathrm{Map}_R G(S, \mathrm{Map}_{\mathbf{Sp}}(S[G], S[G])) \\
\simeq &\mathrm{Map}_R G(S[G], S[G]) \\
\simeq &S[G]
\end{aligned}$$

(just believe me here, you don't want to check the G-actions by yourself). □

Corollary 2.0.8. *Dually we define $S^{-\mathrm{ad}G} = (DG_+)_{hLG}$. Moreover, we have $S[G] \otimes S^{-\mathrm{ad}G} \simeq DG_+$.*

Corollary 2.0.9. *This implies $S^{\mathrm{ad}G} \otimes S^{-\mathrm{ad}G} \simeq S$.*

Proof. From the above we deduce that $S[G] \otimes S^{\mathrm{ad}G} \otimes S^{-\mathrm{ad}G} \simeq S[G]$. Then by shearing the G actions on both sides to only acting on $S[G]$ if necessary, taking G-homotopy orbits on both sides we obtain the result. □

Corollary 2.0.10. *For X a left G-module, this map induces the norm map $N : (X \otimes S^{\mathrm{ad}G})_{hG} \to X^{hG}$. As a result, we may define the Tate construction X^{tG} to be the cofiber of the norm map.*

Proof. Consider the limit-colimit exchange map

$$\kappa : ((X \otimes S[G])^{hRG})_{hLG} \to ((X \otimes S[G])_{hLG})^{hRG}$$

where $X \otimes S[G]$ is equipped with the right action acting only on $S[G]$, and the left action acting diagonally. Then by definition, $(X \otimes S[G])^{hRG} \simeq X \otimes S^{\mathrm{ad}G}$, and the right hand side is equivalent to X^{hRG} by the shear map $X \otimes S[G] \to X \otimes S[G]$ sending the diagonal action to the action only on $S[G]$, and use the fact that $S[G]_{hLG} = S \otimes_G S[G] = S[G]$. Finally, after shearing, we have induced a non-trivial right G action on X, hence we have on the right the X^{hRG} equivalent to X^{hLG} equipped with the original left action. □

Lemma 2.0.11. *We have $(S[G] \otimes X)^{tG} \simeq *$.*

Proof. Notice the source of the norm map can be identified with $(S[G] \otimes X \otimes S^{\mathrm{ad}G})_{hLG} \simeq X \otimes S^{\mathrm{ad}G}$, since we may always shear the G-action on the left to the action only on $S[G]$. Then on the right we have

$$(S[G] \otimes X)^{hRG} \simeq (DG_+ \otimes S^{\mathrm{ad}G} \otimes W)^{hRG} \simeq \mathrm{Map}(S[G], W \otimes S^{\mathrm{ad}G})^{hRG} \simeq W \otimes S^{\mathrm{ad}G}$$

Then we obtain the result. □

With these preliminaries, we may begin our topic on Galois extensions. Moreover, from now on, all G actions are left G actions, so we discard the notation hRG and use hG for hLG.

Chapter 3

Galois extensions of ring spectra

Definition 3.0.1. Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let G be a stably dualizable group acting on the left B through $\mathbb{E}_\infty$ A-algebras (this basically implies that $A \to B$ is also a map of left G-modules). We say that this is a G-Galois extension if we have $A \to B^{hG}$ is an equivalence, and $h : B \otimes_A B \to \mathrm{Map}(S[G], B)$ is an equivalence, where this second map is obtained from the dual $B \otimes_A B \otimes S[G] \to B \otimes_A B \to B$.

The definition is well enough intuitive, and we present a first example of G-Galois extensions.

Lemma 3.0.2. *Let $R \to T$ be a homomorphism of discrete commutative rings, and let G be a finite group acting on T through R-algebra homomorphisms. Then $R \to T$ is a G-Galois extension iff $HR \to HT$ is a G-Galois extension.*

Proof. $\Longrightarrow$: A theorem states that T is a finitely generated projective R-module, and T is moreover invertible as an $R[G]$-module. Therefore T is also flat, and hence $\mathrm{Tor}_s^R(T, T) = 0$ for $s \neq 0$. Furthermore, we have $\mathrm{Ext}^s_{R[G]}(R, R[G]) = \mathrm{Ext}^s_R(R, R) = 0$ for $s \neq 0$ by a Grothendieck spectral sequence argument, hence we have $\mathrm{Ext}^s_{R[G]}(R, T) = 0$ for $s \neq 0$. Therefore we apply the spectral sequences

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(M_*, N_*) \Longrightarrow \mathrm{Tor}^R_{p+q}(M, N)$$
$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{R_*}(M^*, N^*) \Longrightarrow \mathrm{Ext}^{p+q}_R(M, N)$$

to the above and obtain spectral sequences

$$E^2_{p,q} = H^p(G; \pi_{-q}HT) = \mathrm{Ext}^{p,q}_{LR[G]}(R, T) \Longrightarrow \pi_{s+t}(HT^{hG})$$

$$E^2_{p,q} = \mathrm{Tor}^R_{p,q}(T, T) \Longrightarrow (HT \otimes_{HR} HT)$$

Notice by the above argument, the spectral sequences collapse and we obtain $(HT)^{hG} \simeq H(T^G) = HR$, and $HT \otimes_{HR} HT \simeq H(T \otimes_R T) \simeq H \prod_G T \simeq \prod_G HT$. Thus $HR \to HT$ is a G-Galois extension of ring spectra.
The converse is obvious. $\square$

Unlike the case of fields, since we are working with ring spectra, we are allowed to have trivial extensions.

Lemma 3.0.3. *For any $\mathbb{E}_\infty$-ring spectrum A with G stably dualizable, the map $A \to \mathrm{Map}(S[G], A)$ is a G-Galois extension. The map $A \to \mathrm{Map}(S[G], A)$ is the dual of the collapse map mapping G to $*$.*

Proof. Direct. $\square$

The above extensions are in fact, faithful. This is in fact, easy to prove, after we have given some definitions and criteria in the following section.

Another example is the well-know fact on K-theory.

Theorem 3.0.4. *Recall that the Adams operation $\psi^{-1} : KU \to KU$ is the operation sending a vector bundle to its complex conjugate, and $(\psi^{-1})^2 = \mathrm{id}$. Then let $c : KO \to KU$ be the complexification, then we have c and ψ^{-1} exhibits KU as a C_2-Galois extension of KO.*

Chapter 4

Dualizability and faithfulness

We first give some lemmas on dualizability and faithfulness.

Definition 4.0.1. Recall for A an $\mathbb{E}_\infty$-algebra, an A-module M is faithful if for any other A-module N we have $M \otimes_A N = 0 \implies N = 0$. We say an A-algebra B is faithful over A if it is faithful as an A-module.

Recall the fact that we have for discrete rings that all Galois extensions are faithful. However, this is not the case for ring spectra, and we will later present many criteria for an extension of ring spectra to be faithful. The main reason that we care about faithful extensions is that the Galois correspondence only works for faithful extensions.

Lemma 4.0.2. *Let $A \to B$ be a map of $\mathbb{E}_\infty$-algebras, and M a faithful A-module, we have the following.*

1. *A map of A-modules $X \to Y$ is a weak equivalence iff $M \otimes X \to M \otimes Y$ is a weak equivalence.*
2. *$B \otimes_A M$ is a faithful B-module.*
3. *If $A \to B$ is moreover faithful, and for some A-module N we have $B \otimes_A N$ is a faithful B-module we have N is a faithful A-module.*
4. *For $R \to T$ a faithfully flat map of discrete rings, we have $HR \to HT$ a faithful map of $\mathbb{E}_\infty$-ring spectra.*

Proof. The first assertion is direct by considering the mapping cone of $X \to Y$.
The second assertion is by the associativity of the smash product.
For the third property, consider some A-module X. Then if $X \otimes_A N = 0$, we have $(X \otimes_A B) \otimes_B (B \otimes_A N) = (X \otimes_A N) \otimes_A B = 0$, hence $X \otimes_A B = 0$, but since B is faithful over A we have $X = 0$.
Finally for the last property, consider an HR-module N. Then we have the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^R_{p,q}(\pi_*(N), T) \Longrightarrow \pi_{p+q}(N \otimes_{HR} HT)$$

which collapses at the first page by flatness of T. Thus we have $\pi_*(N) \otimes_R T = 0$, hence by faithfulness of T we have $\pi_*(N) = 0$, hence $N = 0$. □

Lemma 4.0.3. *Let $A \to B$ be a map of $\mathbb{E}_\infty$-algebras, and M an A-module. Then we have the following.*

1. *If M is dualizable as an A-module, then $B \otimes_A M$ is a dualizable B-module.*
2. *If $A \to B$ is faithful, B dualizable over A, and $M \otimes_A B$ is a dualizable B-module, then M is a dualizable A-module.*

Proof. For the first assertion, we verify the map $\mathrm{Map}_B(B\otimes_A M, B)\otimes_B (B\otimes_A M) \to \mathrm{Map}(B\otimes_A M, B\otimes_A M)$ is an equivalence. But this is direct by

$$\begin{aligned} &\mathrm{Map}_B(B\otimes_A M, B)\otimes_B (B\otimes_A M) \\ \simeq &\mathrm{Map}_A(M, B)\otimes_A M \\ \simeq &\mathrm{Map}_A(M, B\otimes_A M) \\ \simeq &\mathrm{Map}_B(B\otimes_A M, B\otimes_A M) \end{aligned}$$

using the fact that M is dualizable as an A-module.
For the second assertion, to show $\mathrm{Map}_A(M, A)\otimes_A M \simeq \mathrm{Map}_A(M, M)$, it suffices to show that we have an equivalence after tensoring B on both sides using that B is faithful over A. But notice that we have $B\otimes_A \mathrm{Map}_A(M, A) \simeq \mathrm{Map}_A(M, B)$ by faithfulness of B, and similarly $\mathrm{Map}_A(M, M)\otimes_A B \simeq \mathrm{Map}_A(M, M\otimes_A B)$. Hence the desired map is an equivalence iff the following map is an equivalence

$$\mathrm{Map}_B(B\otimes_A M, B)\otimes_B (B\otimes_A M) \to \mathrm{Map}_B(B\otimes_A M, B\otimes_A M)$$

which is by definition. $\square$

Lemma 4.0.4. *Let $A\to B$ be a map of $\mathbb{E}_\infty$-ring spectra, and let G be a stably dualizable group acting on B through A-algebra homomorphisms. Then if M is a dualizable A-module, then the canonical map $M\otimes_A B^{hG}\simeq (M\otimes_A B)^{hG}$ is an equivalence.*

Proof. Notice that the left hand side could be identified with $\mathrm{Map}(D_A M, B^{hG})$, while the right hand side could be identified with $\mathrm{Map}_A(D_A M, B)^{hG}$, and we may simply use the fact that taking limits commutes with the second factor of Map. $\square$

Proposition 4.0.5. *Let A and B be $\mathbb{E}_\infty$ ring spectra, and B be a G-Galois extension of A, then we have*

1. *For each B-module M a natural equivalence*

$$h_M : M\otimes_A B \to \mathrm{Map}_{\mathbf{Sp}}(S[G], M)$$

2. *For each B-module M a natural equivalence*

$$j_M : M\otimes S[G] \to \mathrm{Map}_A(B, M)$$

Proof. We define h_M to be the composite

$$M\otimes_A B \simeq M\otimes_B B\otimes_A B \simeq M\otimes_B \mathrm{Map}(S[G], B) \to \mathrm{Map}(S[G], M)$$

This is a weak equivalence since in the last map $S[G]$ is dualizable, hence $\mathrm{Map}(S[G], X)\simeq DG_+\otimes X$.
Then we give the definition of j_M, prove the second assertion. We define j_M again through its adjoint

$$M\otimes G_+\otimes_A B \xrightarrow{1\otimes m} M\otimes_A B \to M$$

Then we have

$$\begin{aligned} j_M : &M\otimes S[G] \\ &\simeq \mathrm{Map}(DG_+, M) \\ &\simeq \mathrm{Map}_B(B\otimes DG_+, M) \\ &\simeq \mathrm{Map}_B(\mathrm{Map}(G_+, B), M) \\ &\to \mathrm{Map}_B(B\otimes_A B, M) \\ &\simeq \mathrm{Map}_A(B, M) \end{aligned}$$

where the only arrow is an equivalence for the first assertion. $\square$

Proposition 4.0.6. *Let $A \to B$ be a G-Galois extension, then B is a dualizable A-module.*

Proof. We show the map $\mathrm{Map}_A(B, A) \otimes B \to \mathrm{Map}_A(B, B)$ is an equivalence. Note that this is the top map in the following diagram.

$$\begin{tikzcd}
M \otimes_A \mathrm{Map}_A(B, A) \arrow[r] \arrow[d] & \mathrm{Map}_A(B, M \otimes_A A) \arrow[d] \\
M \otimes_A \mathrm{Map}_A(B, B^{hG}) \arrow[r] \arrow[d] & \mathrm{Map}_A(B, M \otimes_A B^{hG}) \arrow[d] \\
M \otimes_A \mathrm{Map}(B, B)^{hG} \arrow[r] & \mathrm{Map}_A(B, M \otimes_A B)^{hG} \\
M \otimes_A (B \otimes S[G])^{hG} \arrow[r] \arrow[u] & (M \otimes_A B \otimes S[G])^{hG} \arrow[u] \\
M \otimes_A (B \otimes S[G] \otimes S^{\mathrm{ad}G})_{hG} \arrow[r] \arrow[u] & (M \otimes_A B \otimes S[G] \otimes S^{\mathrm{ad}G})_{hG} \arrow[u]
\end{tikzcd}$$

Since M has no G-action on it, the bottom map is an equivalence. Then the top left map is an equivalence by definition, the second left map is an equivalence since limits commutes with $\mathrm{Map}(X, -)$, and the third left map is an equivalence by the preceding proposition, and the bottom left map is an equivalence by the lemma we proved after we defined the norm map. Therefore, all the maps in the diagrams are equivalences. $\square$

Then, we present a few criteria for faithful extensions.

Proposition 4.0.7. *A G-Galois extension $A \to B$ is faithful iff the norm map $N : (B \otimes S^{\mathrm{ad}G})_{hG} \to B^{hG}$ is an equivalence.*

Proof. If the norm map is an equivalence, and M is an A-module such that $M \otimes_A B \simeq *$, then we have the following.

$$M \simeq M \otimes_A A \simeq M \otimes_A B^{hG} \simeq M \otimes_A (B \otimes S^{\mathrm{ad}G})_{hG} \simeq (M \otimes_A \otimes_B \otimes S^{\mathrm{ad}G})_{hG} \simeq *$$

Thus $A \to B$ is faithful.
For the converse, it suffices to prove $B \otimes_A (B \otimes S^{\mathrm{ad}G})_{hG} \to B \otimes_A B^{hG}$ is an equivalence. To prove this, consider the diagram

$$\begin{tikzcd}
B \otimes_A (B \otimes S^{\mathrm{ad}G})_{hG} \arrow[r] \arrow[d] & (B \otimes_A B \otimes S^{\mathrm{ad}G})_{hG} \arrow[r] \arrow[d] & (\mathrm{Map}(S[G], B) \otimes S^{\mathrm{ad}G})_{hG} \arrow[d] \\
B \otimes_A B^{hG} \arrow[r] & (B \otimes_A B)^{hG} \arrow[r] & \mathrm{Map}(S[G], B)^{hG}
\end{tikzcd}$$

Notice that all the horizontal maps are equivalences by preceding lemmas or definitions, and the rightmost vertical map is an equivalence by a dual argument of the fact that the norm map is an equivalence on free G-modules. $\square$

Proposition 4.0.8. *Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let G be a stably dualizable group acting on B through A-algebra maps. Then $A \to B$ is a faithful G-Galois extension iff $B \otimes_A B \to \mathrm{Map}(S[G], B)$ is an equivalence and B is faithful and dualizable over A.*

Proof. $\implies$ is obvious, and we prove $\impliedby$. We show $A \to B^{hG}$ hence $B \to B \otimes_A B^{hG}$ is an equivalence. But we have $B \simeq \mathrm{Map}(S[G], B)^{hG} \simeq (B \otimes_A B)^{hG} \simeq B \otimes_A B^{hG}$ where the last map is again by the dualizability of B. $\square$

Proposition 4.0.9. *Let $A \to B$ be a G-Galois extension with G an $\mathbb{E}_\infty$ group. Then B is smash invertible as an $A[G]$-module iff $A \to B$ is faithful.*

Proof. $\Longrightarrow$: On this side G need not to be Abelian. Suppose the smash inverse of B is C, that is, $B \otimes_{A[G]} C \simeq A[G]$, and $N \otimes_A B \simeq *$, then $N \otimes_A A[G] \simeq N \otimes_A B \otimes_{A[G]} C \simeq *$, hence N, as a retract of $N \otimes_A A[G]$, is contractible.
$\Longleftarrow$: We prove that the smash inverse of B is its dual $D_{A[G]}B = \mathrm{Map}_{A[G]}(B, A[G])$. In fact, we prove the evaluation map $\mathrm{Map}_{A[G]}(B, A[G]) \otimes_{A[G]} B \to A[G]$ is an equivalence. Since $A \to B$ is faithful, and that both sides are endowed with the natural A-module structures, hence it suffices to prove that

$$B \otimes_A \mathrm{Map}_{A[G]}(B, A[G]) \otimes_{A[G]} B \to B[G]$$

is an equivalence. Then this map factors as the following

$$\begin{aligned} &B \otimes_A \mathrm{Map}_{A[G]}(B, A[G]) \otimes_{A[G]} B \\ \simeq &\mathrm{Map}_{A[G]}(B, B[G]) \otimes_{A[G]} B \\ \simeq &\mathrm{Map}_{B[G]}(B \otimes_A B, B[G]) \otimes_{B[G]} (B \otimes_A B) \\ \to &B[G] \end{aligned}$$

where the first equivalence is by the dualizability of B, and the second equivalence is natural. Hence it suffices to prove that the last evaluation map is an equivalence. Then, we have an equivalence of $B[G]$-module maps (since G is Abelian, we don't distinguish between left and right modules here).

$$(B \otimes_A B) \otimes S^{\mathrm{ad}G} \simeq \mathrm{Map}(S[G], B) \otimes S^{\mathrm{ad}G} \simeq B \otimes DG_+ \otimes S^{\mathrm{ad}G} \simeq B[G] \to B[G]$$

Where the last map is given by the inverse map of G (actually I believe that this is the place where the $\mathbb{E}_\infty$-structure of G comes in). Therefore by replacing $B \otimes_A B$ by $B[G]$ through this (shifted) equivalence in the above map, it suffices to prove that $\mathrm{Map}_{B[G]}(B[G], B[G]) \otimes_{B[G]} (B[G]) \to B[G]$ is an equivalence, which is obvious. □

Lemma 4.0.10. *Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let $A \to C$ be a G-Galois extension. Then we have the following.*

1. *If $A \to C$ is faithful, then $B \to B \otimes_A C$ is a faithful G-Galois extension.*
2. *If B is dualizable over A, then $B \to B \otimes_A C$ is a G-Galois extension.*

Proof. Notice the action $G \curvearrowright C$ extends to $G \curvearrowright B \otimes_A C$ by sending $g : C \to C$ to $1 \otimes C : B \otimes_A C \to B \otimes_A C$. For the first assertion, notice that C is dualizable and faithful over A by assumption, hence $B \otimes_A C$ is dualizable and faithful over B by the lemmas at the beginning of this section. Finally we verify $(B \otimes_A C) \otimes_B (B \otimes_A C) \to \mathrm{Map}(S[G], B \otimes_A C)$ is an equivalence. By definition, this map is equivalent to the map $B \otimes_A C \otimes_A C \to B \otimes_A \mathrm{Map}(S[G], C)$, which is clearly an equivalence since C is G-Galois over A.
For the second assertion, the map $(B \otimes_A C) \otimes_B (B \otimes_A C) \to \mathrm{Map}(S[G], B \otimes_A C)$ is an equivalence by exactly the same argument as above, and $B \to (B \otimes_A C)^{hG}$ is an equivalence since $B \simeq B \otimes_A A \simeq B \otimes_A C^{hG} \simeq (B \otimes_A C)^{hG}$ since B is dualizable over A. □

Lemma 4.0.11. *Let $A \to B$ and $A \to C$ be maps of $\mathbb{E}_\infty$ ring spectrum, and suppose B is faithful and dualizable over A. Then let G be a stably dualizable group acting on C through A-algebra maps. Then we have $B \to B \otimes_A C$ is a (faithful) G-Galois extension implies $A \to C$ is a (faithful) G-Galois extension.*

Proof. We verify that $A \to C^{hG}$ and $C \otimes_A C \to \mathrm{Map}(S[G], C)$ are equivalences. For the left, notice that $B \simeq B \otimes_A A \to B \otimes_A C^{hG} \simeq (B \otimes_A C)^{hG}$ is an equivalence by the assumption, and by faithfulness of B we obtain the result. The second map is an equivalence since $B \otimes_A C \otimes_A C \to B \otimes_A \mathrm{Map}(S[G], C)$ is an equivalence by the same reason as above. □

Chapter 5

The trace map and the discriminant map

Definition 5.0.1. For $A \to B$ a G-Galois extension of $\mathbb{E}_\infty$ ring spectra, we define the trace map (as an A-module map)

$$\mathrm{tr} : B \otimes S^{\mathrm{ad}G} \to A$$

by the following

$$B \otimes S^{\mathrm{ad}G} \twoheadrightarrow (B \otimes S^{\mathrm{ad}G})_{hG} \xrightarrow{N} B^{hG} \simeq A$$

or equivalently the following

$$B \otimes S^{\mathrm{ad}G} \simeq B \otimes S[G]^{hG} \simeq (B \otimes S[G])^{hG} \to B^{hG} \simeq A$$

where the map $(B \otimes S[G])^{hG} \to B^{hG}$ is derived from the homotopy fixed point of the map of G acting on the right of B via the inverse of the left action.

When G is finite, we have a better characterization of the trace map by the following.

Lemma 5.0.2. *For $A \to B$ a G-Galois extension with G finite, then the composite*

$$B \xrightarrow{\mathrm{tr}} A \to B$$

is the sum of all the actions of $g \in G$ on B, and the composite

$$A \to B \xrightarrow{\mathrm{tr}} A$$

is the multiplication by $|G|$ map.

Proof. This is by directly calculating norm map. □

Corollary 5.0.3. tr *is a split surjective map of A-modules if $|G|$ is invertible in $\pi_0(A)$, hence such extension is faithful.*

Proof. By definition. □

Definition 5.0.4. We define the trace pairing $B \otimes_A B \otimes S^{\mathrm{ad}G} \to A$ as a map of A-modules is defined as the composite

$$B \otimes_A B \otimes S^{\mathrm{ad}G} \to B \otimes S^{\mathrm{ad}G} \to A$$

Then as in the classical case, we define the discriminant map $\mathrm{disc}_{B/A} : B \otimes S^{\mathrm{ad}G} \to D_A B$ as the following composition.

$$\begin{aligned}
&B \otimes S^{\mathrm{ad}G} \\
\simeq &B \otimes S[G]^{hG} \\
\simeq &(B \otimes S[G])^{hG} \\
\xrightarrow{\zeta} &(B \otimes S[G])^{hG} \\
\simeq &\mathrm{Map}_A(B, B)^{hG} \\
\simeq &\mathrm{Map}_A(B, B^{hG}) \\
\simeq &D_A B
\end{aligned}$$

where ζ is the shearing map sending the G action solely on $S[G]$ to the diagonal action on $B \otimes S[G]$, hence this can be identified with the twisted group ring.

Lemma 5.0.5. *We have* $\mathrm{disc}_{B/A} : B \otimes S^{\mathrm{ad}G} \to D_A B$ *right adjoint to the trace pairing* $B \otimes_A B \otimes S^{\mathrm{ad}G} \to A$*.*

Proof. This is direct, but we should use the second definition of the trace to see this better. □

Proposition 5.0.6. *For* $A \to B$ *a* G*-Galois extension, we have* $\mathrm{disc}_{B/A} : B \otimes S^{\mathrm{ad}G} \to D_A B$ *an weak equivalence. In particular,* B *is self-dual as an* A*-module taking into account a tensor invertible* $S^{\mathrm{ad}G}$*.*

Proof. By definition, all maps used to define $\mathrm{disc}_{B/A}$ are equivalences. □

Chapter 6

The Galois correspondence

We first address the easy part of the Galois correspondence. When our Galois groups are finite groups, we considered the fixed subfield of its finite subgroups. But now, our groups have been generalized to stably dualizable groups, hence we also need to generalize the finite subgroups to the following called allowable subgroups.

Definition 6.0.1. Let G be a stably dualizable group, and we say a subgroup $K \subseteq G$ is allowable if K is stably dualizable, the collapse map $G \times_K EK \to G/K$ is a stable equivalence (which says that the quotient behaves homotopically), and the projection $G \to G/K$ admits a section. We say K is an allowable normal subgroup if furthermore K is normal.
Finally we define the equivalences between allowable subgroups to be the minimal equivalence relation generated by the inclusions $K \subseteq K'$ and $K \hookrightarrow K'$ is a stable equivalence.

We remark here that this is the correct notion of subgroups in our context.

Theorem 6.0.2. *Let $A \to B$ be a faithful G-Galois extension and $K \subseteq G$ be any allowable subgroup. Then the inclusion $C = B^{hK} \to B$ is a faithful G-Galois extension. Moreover, if K is an allowable normal subgroup, then $A \to C$ is a faithful G-Galois extension.*

Proof. We will prove the extensions are Galois by detecting it through the dualizable and faithful algebras. That is, since B is faithful and dualizable over A, we have $C \otimes_A B$ is faithful and dualizable over C. Thus to prove $C \to B$ is Galois, it suffices to prove that

$$B \otimes_A C = C \otimes_C (C \otimes_A B) \to B \otimes_C (C \otimes_A B) = B \otimes_A B$$

is Galois. We consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[d] & B \arrow[l] \arrow[d] \\
C \arrow[r] \arrow[d] & C \otimes_A B \arrow[r] \arrow[d] & \mathrm{Map}(S[G], B)^{hK} \arrow[d] & \mathrm{Map}(\Sigma^\infty_+ G/K, B) \arrow[l] \arrow[d] \\
B \arrow[r] & B \otimes_A B \arrow[r] & \mathrm{Map}(S[G], B) & \mathrm{Map}(S[G], B) \arrow[l]
\end{tikzcd}$$

we must prove that all the horizontal maps on the middle and right are equivalences, so that the extension $B \otimes_A C \to B \otimes_A B$ would be equivalent to the extension $\mathrm{Map}(\Sigma^\infty_+ G/K, B) \to \mathrm{Map}(S[G], B)$. However, by the allowable condition, we would have

$$\mathrm{Map}(S[G], B) \simeq \mathrm{Map}(\Sigma^\infty_+(K \times G/K), B) \simeq F(S[K], \mathrm{Map}(\Sigma^\infty_+(G/K), B))$$

which is equivalent to the trivial extension. In particular, since the trivial extension is faithful, this is also faithful. Now, only the horizontal maps in the second row are non-trivially equivalences. $B \otimes_A C \to \mathrm{Map}(S[G], B)^{hK}$ is by definition the composite

$$B \otimes_A C \simeq B \otimes_A B^{hK} \simeq (B \otimes_A B)^{hK} \simeq \mathrm{Map}(S[G], B)^{hK}$$

hence is an equivalence.
The map $\mathrm{Map}(\Sigma^\infty_+(G/K), B) \to \mathrm{Map}(S[G], B)$ is an equivalence since it is the composite

$$\mathrm{Map}(\Sigma^\infty_+(G/K), B) \simeq \mathrm{Map}(S[G]_{hK}, B) \simeq \mathrm{Map}(S[G], B)^{hK}$$

by the allowability condition.
Finally it remains to treat the case where K is normal. If this is the case, then G/K is also a group, hence the extension $B \to B \otimes_A C$ is equivalent to the trivial extension $B \to \mathrm{Map}(S[G/K], B)$, hence is faithfully G/K-Galois. And since B is faithful and dualizable over A, we may reflect this faithful extension back and obtain that $A \to C$ is faithfully G/K-Galois. □

The converse is much more difficult. Firstly, we cannot do the the converse result for the fully general case for calculational reasons, and this actually involves a deeply involved calculational technique developed by Goerss-Hopkins. We give a brief introduction to this theory, and apply it to the converse problem.

6.1 Goerss-Hopkins obstruction theory

We first give our the underlying category of our theory.

Definition 6.1.1. We define a category $\mathcal{C}$ to be Grothendieck prestable if there is some presentable stable category $\mathcal{D}$ equipped with a t-structure compatible with filtered colimits, and $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$. In fact, we can canonically find such $\mathcal{D}$ by taking $\mathcal{D} = \mathbf{Sp}(\mathcal{C})$, and associate it with the t-structure coming from $\mathcal{C}$.
If furthermore $\mathcal{C}$ is endowed with a symmetric monoidal structure, we require the above definitions to be compatible with our symmetric monoidal structure.

Definition 6.1.2. Notice for such $\mathcal{C}$, we have doing truncations and taking connective covers agreeing with the t-structure of $\mathcal{D}$. Then we define $\mathcal{C}^\heartsuit$ by its discrete objects, and we have $\mathcal{C}^\heartsuit \simeq \mathbf{Sp}(\mathcal{C})^\heartsuit$. Then we define for $X \in \mathcal{C}$, $\pi_i(X) \in \mathcal{C}^\heartsuit$ to be $(\Omega_i X)_{\leq 0}$.
Notice that a symmetric monoidal structure on $\mathcal{C}$ gives a symmetric monoidal structure on $\mathcal{C}^\heartsuit$ making $\pi_0 : \mathcal{C} \to \mathcal{C}^\heartsuit$ symmetric monoidal.

Definition 6.1.3. We define a grading on a symmetric monoidal Grothendieck prestable category $\mathcal{C}$ to be an autoequivalence $[1] : \mathcal{C} \to \mathcal{C}$ requiring $(c \otimes d)[1] \simeq c[1] \otimes d$. As usual, we will denote by its inverse $[-1]$.
The autoequivalence preserves Σ and Ω, and hence preserves the t-structure of $\mathcal{C}$. In particular, $[1]$ preserves the subcategory of discrete objects, so $[1]$ restricts to an autoequivalence $\mathcal{C}^\heartsuit \to \mathcal{C}^\heartsuit$, and we have $(\pi_i X)[k] \simeq \pi_i(X[k])$. We define the bigraded Ext groups for objects in $\mathcal{C}^\heartsuit$ by

$$\mathrm{Ext}^{s,t}_{\mathcal{C}^\heartsuit}(A, B) = \mathrm{Hom}_{D(\mathcal{C}^\heartsuit)}(A, \Sigma^s B[-t])$$

viewed as an object in $\mathcal{C}^\heartsuit$. Finally we can always give any Grothendieck prestable category the trivial grading by $[1] = \mathrm{id}_{\mathcal{C}}$.

Definition 6.1.4. We say $\mathcal{C}$ is separable if the homotopy groups detect equivalences, and complete if the Postnikov towers converge, that is, for every $X \in \mathcal{C}$ we have $X \to \varprojlim_n X_{\leq n}$ is an equivalence.

Definition 6.1.5. A shift algebra A is an $\mathbb{E}_1$-algebra in $\mathcal{C}$ equipped with a map $\tau : \Sigma A[-1] \to A$ of right A-modules inducing an isomorphism $\pi_* A \simeq \pi_0 A[\tau]$, where the latter is defined by the graded algebra in $\mathcal{C}^\heartsuit$ by $(\pi_0 A[\tau])_k \simeq (\pi_0 A)[-k]$.

Lemma 6.1.6. *Let A be a shift algebra, and let M be a left A-module. Then the following are equivalent.*

1. $\pi_* M \simeq \pi_* A \otimes_{\pi_0 A} \pi_0 M$*.*

2. $\pi_0 A \otimes_A M \in \mathcal{C}^{\heartsuit}$*.*

3. $\tau \otimes \mathrm{id}_M : \Sigma M[-1] \to M$ *is a 1-connective cover.*

Proof. Notice that τ induces a cofiber sequence of right A-modules

$$\Sigma A[-1] \to A \to \pi_0 A$$

By tensoring with M we obtain

$$\Sigma M[-1] \to M \to \pi_0 A \otimes_A M$$

which implies that (2) $\iff$ (3). Then notice that (1) $\implies$ (3) since $\pi_k M = \tau^k \pi_0 M[-k]$, hence taking $k = 1$ we obtain condition (3). Finally if (2) $\iff$ (3) holds, we have $\pi_0 M \simeq \pi_0 A \otimes_A M$ and $\pi_k M = \pi_1 A \otimes \pi_{k-1} M$ by the long exact sequence associated to the above cofiber sequence, which clearly proves condition (1). $\square$

Definition 6.1.7. We define a left A-module M to be periodic if it satisfies the above equivalent conditions. We denote the category of periodic modules to be the full subcategory $\mathbf{Mod}_A^{per}(\mathcal{C}) \subseteq \mathbf{Mod}_A(\mathcal{C})$.

The category of periodic modules, under mild conditions, is stable. Explicitly we have

Theorem 6.1.8. *Let A be a shift $\mathbb{E}_2$-algebra, then we have $\mathbf{Mod}_A^{per}(\mathcal{C}) \simeq \mathbf{Mod}_A^{\tau^{-1}}(\mathbf{Sp}(\mathcal{C}))$, where the second category is the stable subcategory of $\mathbf{Mod}_A(\mathbf{Sp}(\mathcal{C}))$ consisting of τ-local modules.*

It might seem strange at this point, but what we are trying to do is actually find methods to recover periodic modules over shift algebras, which will finally lead to a spectral sequence calculating (in our setting) $\mathrm{Hom}_{\mathbb{E}_\infty - A - \mathbf{Alg}}(B, B)$. We will do the first thing by building the periodic module layer by layer. Hence from now on, we assume that the base Grothendieck prestable category to be complete. Therefore, we will abuse notation by writing $A_{\leq \infty} = A$.

Construction 6.1.9. *For any associative algebra A, its postnikov tower is a tower of associative algebras*

$$A \to \cdots \to A_{\leq n} \to \cdots \to A_{\leq 0}$$

where the passage from $A_{\leq n-1}$ to $A_{\leq n}$ is obtained by a square zero extension (we just point it out here, and it would be crucial for later purposes). This induces a tower of Grothendieck prestable categories by

$$\mathbf{Mod}_A \to \cdots \to \mathbf{Mod}_{A_{\leq n}} \to \cdots \to \mathbf{Mod}_{A_{\leq 0}}$$

By giving a filtration on each category $\mathbf{Mod}_{A_{\leq n}}$ by truncation, we may prove that $\mathbf{Mod}_A \simeq \varprojlim \mathbf{Mod}_{A_{\leq n}}$. Hence to understand $\mathbf{Mod}_A$, it suffices to understand $\mathbf{Mod}_{A_{\leq 0}}$, and the relations between $\mathbf{Mod}_{A_{\leq n}}$ and $\mathbf{Mod}_{A_{\leq n-1}}$.

For the second problem, we do the following construction.

Construction 6.1.10. *We define $F_n \in \mathbf{Mod}_{A \leq n}$ by the fiber sequences $F_n \to A_{\leq n} \to A_{\leq n-1}$, which gives a pullback diagram of the form*

$$\begin{tikzcd}
A_{\leq n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A_{\leq n-1} \arrow[d, "d_0"] \\
A_{\leq n-1} \arrow[r, "d"'] & A_{\leq n-1} \oplus \Sigma F_n
\end{tikzcd}$$

Here $A_{\leq n-1} \oplus \Sigma F_n$ is a trivial square-zero extension, and d_0 is the trivial section, while d is a map of algebras depending on the square-zero extension $A_{\leq n} \to A_{\leq n-1}$. This square induces a pullback square of module categories.

$$\begin{tikzcd}
\mathbf{Mod}_{A_{\leq n}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbf{Mod}_{A_{\leq n-1}} \arrow[d, "d_0^*"] \\
\mathbf{Mod}_{A_{\leq n-1}} \arrow[r, "d^*"'] & \mathbf{Mod}_{A_{\leq n-1} \oplus \Sigma F_n}
\end{tikzcd}$$

here d^* and d_0^* denote the pushforward (tensor) of modules induced by the map d and d_0. Then we let $\Theta : \mathbf{Mod}_{A_{\leq n-1}} \to \mathbf{Mod}_{A_{\leq n-1}}$ is defined as $\Theta M = d_* d_0^* M$, where d_* is the pullback of modules along d. Explicitly, we have $\Theta M \simeq M \oplus \Sigma^{n+1}(A_{\leq 0} \otimes_{A_{\leq n-1}} M)$.
Then consider the projection $p : A_{\leq n-1} \oplus \Sigma F_n \to A_{\leq n-1}$, noticing that d and d_0 are both sections of p, we have a map $d_* d_0^* M \to d_* p^* p_* d_0^* M \simeq M$. This gives a natural transformation of functors $\pi : \Theta \to \mathrm{id}$.

Theorem 6.1.11. *We define the category of Θ-sections $\Theta - \mathrm{Sect}_{A_{\leq n-1}}$ to be the category of triples (M, s, h), where $M \in \mathbf{Mod}_{A_{\leq n-1}}$, $s : M \to \Theta M$ and h is a homotopy from $\pi \circ s$ to id_M. Then we have $\mathbf{Mod}_{A_{\leq n}} \simeq \Theta - \mathrm{Sect}_{A_{\leq n-1}}$. This means that M can be lifted to $\mathbf{Mod}_{A_{\leq n}}$ iff Θ has a section.*

The first problem is relatively easy.

Theorem 6.1.12. *If $\mathbf{Mod}_A$ is generated by periodic modules, then we have an equivalence of categories*

$$\mathbf{Mod}_A \simeq D_{\geq 0}(\mathbf{Mod}_{\pi_0 A}(C^{\heartsuit}))$$

With these preparations, we will present obstructions to constructing a periodic A-module M with the information of $\pi_0(M)$. In later context, we assume that $\mathbf{Mod}_A$ is generated by periodic A-modules.

Definition 6.1.13. Recall that a module M is periodic iff $\pi_*(M) \simeq \pi_*(A) \otimes_{\pi_0(A)} \pi_0(M)$. We say that an $A_{\leq n}$-module N is a potential n-stage for a periodic A-module if we have $A_{\leq 0} \otimes_{A_{\leq n}} M$ is discrete. We denote the category of potential n-stages by $\mathcal{M}_n$. Equivalently, for $M \in \mathbf{Mod}_{A_{\leq n}}$ and $n < \infty$, we have M is a potential n-stage iff $\pi_*(M) \simeq \pi_*(A_{\leq n}) \otimes_{\pi_0(A)} \pi_0(M)$. This implies that modules of potential n-stages are n-truncated. Moreover, this implies that the A-module structure on M and $A_{\leq n}$-module structure on M is equivalent.

Then we have the following.

Lemma 6.1.14. *Let $M \in \mathbf{Mod}_{A_{\leq n-1}}$. Then $\pi : \Theta M \to M$ is an n-equivalence.*

Proof. Since A is a shift algebra, we have $F_n \simeq \Sigma^n \pi_0 A[-n]$. Then we have $p : A_{\leq n-1} \oplus \Sigma^{n+1} A_{\leq 0}[-n] \to A_{\leq n-1}$ an n-equivalence, and hence so is π since n-equivalences are preserved by forming relative tensor products. □

Corollary 6.1.15. *The cofiber of $\pi : \Theta M \to M$ is equivalent to $\Sigma^{n+2} \pi_0 M[-n]$, using the explicit construction of ΘM.*

Theorem 6.1.16. *Let $M \in \mathcal{M}_{n-1}$, then M could be lifted to a potential n-stage (that is, if there is some $M' \in \mathcal{M}_n$ such that $M \simeq A_{\leq n-1} \otimes_{A \leq n} N$) iff a certain obstruction class $o_M \in \mathrm{Ext}^{n+2,n}_{\pi_0(A)}(\pi_0(M), \pi_0(M))$ vanishes.*

Proof. Recall that M can be lifted to a potential n-stage iff $\Theta M \to M$ has a section, and by using the cofiber sequence $\Theta M \to M \to \Sigma^{n+2}\pi_0 M[-n]$, the first map admits a section iff the second map is 0. Thus the obstruction to the lifting lies in

$$\mathrm{Hom}_{A_{\leq n-1}}(M, \Sigma^{n+2}\pi_0 M[-n]) \simeq \mathrm{Hom}_{\pi_0(A)}(\pi_0(A) \otimes_{A_{\leq n-1}} M, \Sigma^{n+2}\pi_0 M[-n])$$

using the fact that the target is canonically a $\pi_0(A)$-module. By definition that M is a potential $(n-1)$-stage, $\pi_0(A) \otimes_{A_{\leq n-1}} M \simeq \pi_0(M)$, hence the obstruction lies in

$$\mathrm{Hom}_{\pi_0(A)}(\pi_0(M), \Sigma^{n+2}\pi_0(M)[-n]) \simeq \mathrm{Ext}^{n+2,n}_{\pi_0(A)}(\pi_0(M), \pi_0(M))$$

$\square$

This theorem answers the question that whether a $\pi_0(A)$-module can be lifted to a periodic A-module. Moreover, we have a relative version of the theorem calculating the mapping space of two modules from their π_0s. Indeed, we have

Theorem 6.1.17. *Let $M, N \in \mathcal{M}_n$, then let $uM := A_{\leq n-1} \otimes_{A_{\leq n}} M$, we have a fiber sequence*

$$\mathrm{Hom}_{\mathcal{M}_n}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{n-1}}(uM, uN) \to \mathrm{Hom}_{D(\mathbf{Mod}_{\pi_0(A)}(\mathcal{C}^{\heartsuit}))}(\pi_0(M), \Sigma^{n+1}\pi_0 N[-n])$$

Proof. Notice that $\mathcal{M}_n$ is a full subcategory of $\mathbf{Mod}_A$, moreover $uM \simeq M_{\leq n-1}$, hence the left arrow is equivalent to $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_{\leq n-1})$. Letting F be the fiber of $N \to N_{\leq n-1}$, we have a fiber sequence

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_{\leq n-1}) \to \mathrm{Hom}_A(M, \Sigma F)$$

and by noticing $F \simeq \Sigma^n \pi_0 N[-n]$ we obtain the results. $\square$

Corollary 6.1.18. *We have a spectral sequence $E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\pi_0(A)}(\pi_0(M), \pi_0(N))$ converging conditionally to $\pi_* \mathrm{Hom}_A(M, N)$.*

Proof. Notice that $\mathrm{Hom}_A(M, N) \simeq \varprojlim \mathrm{Hom}_{A_{\leq s}}(M \otimes_A A_{\leq s}, N \otimes_A A_{\leq s})$. But by the previous theorem we have the fiber between the filtrations given by $F_s \simeq \mathrm{Hom}_{D(\pi_0(A))}(\pi_0(M), \Sigma^s \pi_0 N[-s])$. Thus we have a spectral sequence associated to this filtration with the required E_1 page. $\square$

Ending up with a spectral sequence computing the mapping space of periodic modules, we are very close to our goal. We then spend some time on generalizing the above part to periodic algebras. Now we assume that A is $\mathbb{E}_\infty$ and $\mathbf{Mod}_A$ is generated by periodic modules.

Definition 6.1.19. A potential n-stage for an $\mathbb{E}_k$-algebra R in $\mathbf{Mod}_{A \leq n}$ such that its underlying $A_{\leq n}$-module is a potential n-stage. We denote the category of potential n-stages for $\mathbb{E}_k$-algebra by $\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)$. This is, of course, equivalent to the condition that $A_{\leq 0} \otimes_{A_{\leq n}} R$ is discrete.
Also by definition $\pi_0(R)$ is an algebra in $\mathbf{Mod}_{\pi_0(A)}(\mathcal{C}^{\heartsuit})$. Finally we have $\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_\infty) \simeq \varprojlim_n \mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)$.

Lemma 6.1.20. *Let R be a potential $(n-1)$-stage for an $\mathbb{E}_k$-algebra. Then the map $\pi : \Theta R \to R$ is a square zero extension of the $\mathbb{E}_k$-$A_{\leq n-1}$-algebra R by the R-module $\Sigma^{n+1}\pi_0 R[-n]$.*

Proof. By definition, since the fiber of π has homotopy concentrated in one positive degree only. $\square$

Construction 6.1.21. *Recall that the $\mathbb{E}_k$-$A_{\leq n-1}$-algebra R has a cotangent complex $\mathbb{L}^{\mathbb{E}_k}_{R/A_{\leq n-1}}$. If you haven't seen this before, just think this as the module of relative Kahler differentials when treating ordinary algebras. As usual, we have an equivalence of the mapping space $\mathrm{Hom}_R(\mathbb{L}^{\mathbb{E}_k}_{R/A_{\leq n-1}}, \Sigma M)$ and the space of square zero-extensions of $\mathbb{E}_k$-$A_{\leq n-1}$-algebras $\tilde{R} \to R$ by M. In particular, an extension is trivial iff its corresponding map from the cotangent complex to ΣM is null.*

Theorem 6.1.22. *Let R be a potential $(n-1)$-stage for an $\mathbb{E}_k$ algebra. Then R can be lifted to a potential n-stage iff some obstruction class*

$$o_R \in \mathrm{Ext}^{n+2,n}_{\mathbf{Mod}_{\pi_0(R)}(D(\mathbf{Mod}_{\pi_0(A)}(\mathcal{C}^\heartsuit)))}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \pi_0(R))$$

vanishes.

Proof. By the previous argument, the obstruction class lives in $\pi_0\mathrm{Hom}_R(\mathbb{L}^{\mathbb{E}_k}_{R/A_{\leq n-1}}, \Sigma^{n+2}\pi_0 R[-n])$. Then this is equivalent to the above group with some calculation since the second term is again concentrated in one degree. □

This theorem provides us obstructions for lifting a commutative or not $\pi_0(A)$-algebra in $\mathcal{C}^\heartsuit$ to $\mathbb{E}_k$-A-algebras. As before, we have the obstructions to lifting a map and the spectral sequence for mapping spaces.

Corollary 6.1.23. *Let R, T be potential n-stages for an $\mathbb{E}_k$-algebra, and let $uR \to uT$ be the corresponding map in the potential $(n-1)$-stages. Then we have an equivalence*

$$F_\varphi \simeq P_{0,\varphi'}\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n+1}\pi_0 T[-n])$$

for F_φ the fiber over φ of the map

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)}(R, T) \to \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_{n-1})}(uR, uT)$$

and φ' is a certain morphism in $\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n+1}\pi_0 T[-n])$, where $P_{0,\varphi'}$ denote the path space.
Finally, we have φ (which lives in the potential $(n-1)$-stage) lifts to the potential n-stage iff $\varphi' \simeq 0$. Moreover, the fiber is equivalent to

$$\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n}\pi_0 T[-n])$$

Proof. Similar to the previous proof. □

Corollary 6.1.24. *Let $\varphi : R \to S$ be a $\mathbb{E}_k$-morphism of periodic A-algebras in $\mathcal{C}$. Then we have a spectral sequence with*

$$E_1^{0,0} = \mathrm{Hom}_{\mathbf{Alg}_{\pi_0(A)}}(\pi_0(R), \pi_0(T))$$

$$E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \pi_0(T)), t \geq s > 0$$

converging conditionally to $\pi_{t-s}(\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathbf{Mod}_A)}(R, T), \varphi)$.

Proof. This is also similar. □

Now, one may wonder how could we remove the periodic conditions on A-algebras, since this is not granted in our settings. To do this, we must pass to another category smartly, and it turns out that we must apply the above theory to the category of connective synthetic spectra.

Construction 6.1.25. *We define an Adams type homology theory E to be a homotopy commutative ring spectrum E which is a filtered colimit $E = \varinjlim E_\alpha$ such that E_*E_α is a finitely generated projective E_*-module, and the Kunneth map $E^*E_\alpha \to \mathrm{Hom}_{E_*}(E_*E_\alpha, E_*)$ is an isomorphism. Then we define $\mathbf{Sp}_E^{fp}$ to be the category of finite spectra with finitely generated projective E-homology, and the covers to be the map inducing E-homology surjections. We define the category of E-synthetic spectra to be $\mathbf{Syn}_E := \mathrm{Sh}^{\mathbf{Sp}}_\Sigma(\mathbf{Sp}_E^{fp})$.*
There is a t-structure on $\mathbf{Syn}_E$. We define for $X \in \mathbf{Syn}_E$, $\pi_n(X)$ to be the sheafification of the presheaf assigning the π_n pointwise. We say X is connective if $\pi_n(X) = 0$ for $n < 0$ and

coconnective if $\Omega^\infty X$ is a discrete sheaf of spaces. In particular, we have $\mathbf{Syn}_E^\heartsuit$ the category consisting of sheaves of Abelian groups. This t-structure is right complete. According to our context, we only care about the connective synthetic spectra.
We have a functor $\nu : \mathbf{Sp} \to \mathbf{Syn}_E$ given by the the sheafification of the connective cover of the Yoneda embedding. Moreover, this functor is lax symmetric monoidal.
Finally, we have a bigrading structure on $\mathbf{Syn}_E$ given by $(\Sigma^{s,t}X)(P) = \Sigma^{-t}X(\Sigma^{-s-t}P)$. The categorical suspension is $\Sigma^{1,-1}$.

With these categorical preliminaries, we may do our work.

Construction 6.1.26. *We give $\mathbf{Syn}_E^{\geq 0}$ a grading by $[1] = \Sigma^{1,0}$. We claim the following. Firstly, we have for an $\mathbb{E}_\infty$ algebra $A \in \mathbf{Sp}_E$, νA an $\mathbb{E}_\infty$ algebra. We have the following.*

1. *νA is a shift algebra.*
2. *$\mathbf{Mod}_{\nu A}^{per} \simeq \mathbf{Mod}_A(Sp_E)$.*
3. *$\mathbf{Mod}_{\pi_0(\nu A)}(\mathbf{Syn}_E^\heartsuit) \simeq \mathbf{Mod}_{E_*A}(\mathbf{Comod}_{E_*E})$.*
4. *$\mathbf{Mod}_{\nu A}$ is generated by the periodic modules.*

We will not give a proof here since we are running out of time. But the above gives rise to our desired spectral sequence.

Theorem 6.1.27. *Let $\varphi : R \to T$ be a morphism of E-local $\mathbb{E}_\infty$-A-algebras, then there is a spectral sequence with*

$$E_1^{0,0} = \mathrm{Hom}_{\mathbf{CAlg}_{E_*A}(\mathbf{Comod}_{E_*E})}(E_*R, E_*T)$$

$$E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\mathbf{Mod}_{E_*R}(\mathbf{CAlg}_{E_*A}(\mathbf{Comod}_{E_*E}))}(\mathbb{L}^{\mathbb{E}_\infty}_{E_*R/E_*A}, E_*T)$$

where E_*T is given E_*R-module structure by the map φ, converging conditionally to

$$\pi_{t-s}(\mathrm{Map}_{\mathbf{CAlg}_A}(R, T), \varphi)$$

Remark. We may clearly apply $\otimes_{E_*A} E_*T$ to the first term in the above, and change the base category to $\mathbf{Alg}_{E_*T}$ and $\mathbf{Mod}_{E_*T}$.

Now, with a few auxillary lemmas, we may give the converse to the Galois correspondence.

Definition 6.1.28. For $A \to B$ a map of $\mathbb{E}_\infty$-ring spectra, we define the topological Hochschild homology of B relative to A by $\mathrm{THH}^A(B) = B \otimes_A S^1$, where this is the tensoring with space in the category $\mathbf{CAlg}(A)$. Equivalently, $\mathrm{THH}^A(B) \simeq B \otimes_{B \otimes_A B} B$. Also equivalently, we have $\mathrm{THH}^A(B)$ is the geometric realization of the cyclic bar construction of B over A. In the last construction, the inclusion into the 0-simplices gives a map $B \to \mathrm{THH}^A(B)$. We say $A \to B$ is THH-etale if $B \to \mathrm{THH}^A(B)$ is an equivalence.
We say $A \to B$ is etale if $\mathbb{L}^{\mathbb{E}_\infty}_{B/A}$ is contractible, B connected if $\pi_0(B)$ is connected, and $A \to B$ separable if $B \otimes_A B \to B$ admits a section.

Lemma 6.1.29. *Every separable $A \to B$ is THH-etale. Notice that with G discrete, the G-Galois extension $A \to B$ is separable, hence is THH-etale.*

Proof. Since G is discrete, we obviously have a retract diagram $B \to B \otimes_A B \to B$, and applying $- \otimes_{B \otimes_A B} B$ we obtain a retract diagram

$$\mathrm{THH}^A(B) \to B \to \mathrm{THH}^A(B)$$

Then we notice that $\mathrm{THH}^A(B) \to B$ given by the multiplication map $B \otimes_A \cdots \otimes_A B \to B$ on each stage of the cyclic bar construction is also a retract. Thus we have $B \simeq \mathrm{THH}^A(B)$. ☐

Theorem 6.1.30. *For any map $A \to B$ of $\mathbb{E}_\infty$-ring spectra, if it is THH-etale, then it is etale.*

Corollary 6.1.31. *Any G-Galois extension $A \to B$ with G discrete is etale. In particular, $\prod_G A$ is etale over A.*

Theorem 6.1.32. *Let $A \to B$ be a G-Galois extension with G finite and B connected. Then the natural map $G \to \mathbf{CAlg}_A(B, B)$ giving the action of G on B is an equivalence of $\mathbb{E}_1$-monoids. In particular, every endomorphism of B over A is an automorphism, up to a contractible choice.*

Proof. We compute $\mathbf{CAlg}_A(B, B) \simeq \mathbf{CAlg}_B(B \otimes_A B, B) \simeq \mathbf{CAlg}_B(\prod_G B, B)$ by the Goerss-Hopkins spectral sequence derived above. We apply the spectral sequence with the case $E = S$. But since $\prod_G B$ is etale over B, we have the spectral sequence collapses complete at the first page leaving only the term $E_1^{0,0} = \mathbf{CAlg}_{B_*}(\prod_G B_*, B_*) \simeq G$ since B_* is connected. Therefore we have the results. ☐

Remark. We notice that it would be a lot more troublesome if G is not discrete.

Theorem 6.1.33. *Let $A \to B$ be a G-Galois extension with B connected and G finite and discrete. Then let $A \to C \to B$ be a factorization of the map through a separable $\mathbb{E}_\infty$-A-algebra C. Then $\mathbf{CAlg}_C(B, B)$ is discrete, and through the map $\mathbf{CAlg}_C(B, B) \to \mathbf{CAlg}_A(B, B)$, $\mathbf{CAlg}_C(B, B)$ can be identified with $K = \pi_0(\mathbf{CAlg}_C(B, B))$ as a subgroup of G. Moreover, the action of K on B induces an equivalence $B \otimes_C B \simeq \prod_K B$.*

Proof. We again apply the Goerss-Hopkins spectral sequence to $\mathbf{CAlg}_C(B, B) = \mathbf{CAlg}_B(B \otimes_C B, B)$, and we again have $E_1^{0,0} \simeq \mathbf{CAlg}_{\pi_*(B)}(\pi_*(B \otimes_C B), \pi_*(B))$, and away from zero we have

$$E_1^{s,t} \simeq \mathrm{Ext}^{2s-t,s}_{\mathbf{Mod}_{\pi_*(B \otimes_C B)}(\mathbf{CAlg}_{\pi_*(B)}(\mathbf{Mod}_{\pi_*(S)}))}(\mathbb{L}^{\mathbb{E}_\infty}_{\pi_*(B \otimes_C B)/\pi_*(B)}, \pi_*(B))$$

Then since $\pi_*(B \otimes_C B)$ is a retract of $\pi_*(B \otimes_A B)$ by applying $B \otimes_C - \otimes_C B$ to the retract diagram $C \to C \otimes_A C \to C$ in the category of $\pi_*(B)$-algebras, we have the cotangent complex in problem a retract of the cotangent complex $\mathbb{L}^{\mathbb{E}_\infty}_{\pi_*(B \otimes_A B)/\pi_*(B)}$, which vanishes since $B \otimes_A B$ is etale over B.
Therefore again $\mathbf{CAlg}_C(B, B)$ is discrete, and $K = \pi_0 \mathbf{CAlg}_C(B, B) = \mathbf{Alg}_{\pi_*(B)}(\pi_*(B \otimes_C B), \pi_*(B))$. Then again using the that $\pi_*(B \otimes_C B)$ is a retract of $\pi_*(B \otimes_A B)$, this implies that K is a retract of G, hence is a submonoid of G. But since G is finite, K must also be a group. Moreover, noticing that $\pi_*(B \otimes_C B)$ is a retract of $\pi_*(B \otimes_A B)$, using the fact that $\pi_*(B)$ is connected, we see that $\pi_*(B \otimes_C B) \simeq \prod_{K'} \pi_*(B)$. Thus since $K = \mathbf{CAlg}_{\pi_*(B)}(\pi_*(B \otimes_C B), \pi_*(B)) \simeq \mathbf{CAlg}_{\pi_*(B)}(\prod_{K'} \pi_*(B), \pi_*(B)) \simeq K'$, hence $K = K'$. Thus we see that $B \otimes_A B \simeq \prod_G B$ restricts to $B \otimes_C B \simeq \prod_K B$. ☐

Corollary 6.1.34. *Let $A \to B$ be a G-Galois extension with B connected and G finite and discrete. Then let $A \to C \to B$ be a factorization of the map through a separable $\mathbb{E}_\infty$-A-algebra C such that $C \to B$ is faithful, and let $K = \pi_0(\mathbf{CAlg}_C(B, B)) \subseteq G$. If $A \to B$ is faithful, then $C \simeq B^{hK}$ and $C \to B$ is a faithful K-Galois extension.*

Proof. We first prove $A \to B$ is faithful implies B dualizable over C. Since C is a retract of $C \otimes_A C$, we have C a dualizable $C \otimes_A C$-module. Moreover, since $C \to B$ is faithful, we have $B \otimes_A C \to B$ exhibits B a dualizable $B \otimes_A C$-module. But $B \otimes_A B$ is a finite direct sum of B, hence $B \otimes_A B$ is also dualizable over $B \otimes_A C$. But since $A \to B$ is dualizable and faithful, we have B dualizable over C.
Moreover we have $B \otimes_C B \simeq \prod_K B$ is a weak equivalence, and since B is faithful and dualizable over C, we conclude that $B \to C$ is a faithful K-Galois extension. ☐

Now we have established our main theorem, and we end the talk with two slight generalizations with two very important examples.

Chapter 7

Generalizations

Definition 7.0.1. Let A be a $\mathbb{E}_\infty$-ring spectra, and let $A \to B_\alpha$ be a directed family of G_α-Galois extensions, such that for $\alpha \leq \beta$, we have $A \to B_\alpha$ is a subextension of $A \to B_\beta$ equipped with a surjection $G_\beta \to G_\alpha$ with kernel a stably dualizable group $K_{\alpha,\beta}$, and the data of a natural equivalence $B_\alpha \simeq B_\beta^{hK_{\alpha,\beta}}$. Let $B = \varinjlim B_\alpha$ where the colimit is taken in $\mathbb{E}_\infty$-A-algebras, and let $G = \varprojlim G_\alpha$, then we say $A \to B$ is a G-pro-Galois extension.

Example. *$L_{K(n)}S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension, where $\mathbb{G}_n$ is the Morava stabilizer group viewed as a profinite group.*

Definition 7.0.2. A Hopf algebra spectrum is a $\mathbb{E}_\infty$-ring spectrum H equipped with a counit $\varepsilon : H \to S$ and a co-$\mathbb{E}_1$ coproduct $\psi : H \to H \otimes H$. We do *not* assume that H has an antipode map.
For $A \to B$ a map of $\mathbb{E}_\infty$-ring spectrum, we define the cobar complex $C^\bullet(H; B)$ with H coacting on B over A to be the cosimplicial commutative A-algebra with

$$C^q(H; B) = B \otimes H^{\otimes q}$$

The coboundary and codegeneracy maps are obviously defined, and let $C(H; B)$ denote its totalization. The algebra unit $A \to B$ induces a coaugmentation $A \to C^\bullet(H; B)$, hence a map $A \to C(H; B)$. We say a map of $\mathbb{E}_\infty$ ring spectrum $A \to B$ is an H-Hopf-Galois extension if H coacts on B over A, so that the maps $A \to C(H; B)$ and $h : B \otimes_A B \to B \otimes H$ are both weak equivalences.

Example. *For an $\mathbb{E}_\infty$-grouplike space X, $S[X]$ is a $\mathbb{E}_\infty$-ring spectrum through the multilplication of G, and is a Hopf algebra spectrum with counit given by the collapse map $X \to *$ and the comultiplication given by the diagonal map $\Delta : X \to X \times X$.*

Example. *The unit map $S \to MU$ is an $S[BU]$-Hopf-Galois extension. This can be viewed as the global version of $L_{K(n)}S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension.*